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Axi-symmetric heat conduction in an anisotropic finite cylinder

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This paper gives unsteadystate temperature distribution in a cylinder of finite length and of finite radius. The conductivity of the material is variable along the direction of the length of the cylinder. The temperature at one end and at the curved surface is prescribed and the initial temperature of the cylinder is known.

After formulating the problem, the resulting partial differential equation is solved with the help of Legendre polynomials, modified Bessel functions and Laplace transform. One special case of boundary conditions is numerically discussed.

1. Introduction

The problems of conduction of heat in anisotropic materials have gained much interest in recent years. These problems occur mainly in wood technology, soil mechanics and mechanics of solids of fibrous structures.

The analytic approach of such problems and their exact solution are hardly available in the present literature. Carslaw & Jaeger (1959) have touched this topic and considered some simple problems. Recently, exact solutions of some problems of anisotropic solids (Saxena 1970, Gupta & Saxena 1971, Saxena & Nagera 1973) have appeared. These problems are concerned with solids of rectangular coordinate system.

In this paper we consider unsteadystate problem of temperature distribution in a cylinder of finite radius and of finite length. The conductivity of the material varies along the direction of the cylinder. Such types of variations of conductivity occur in materials of fibrous structure and also arises from the difference of temperature that exists along the length. The partial differential equation of the problem may be given as

$$k\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}\right)+\frac{\partial}{\partial z}\left(K_z\frac{\partial v}{\partial z}\right)=\rho c\frac{\partial v}{\partial t}, \tag{1}$$

where $v(r, z, t)$ is temperature at any point. r, θ, z are polar cylindrical co-ordinates, z-axis is along the axis of the cylinder a and the origin lies at the centre of one end. K_z is conductivity along the length, k is conductivity along radial directions and ρ and c are density and specific heat respectively.

We assume that the temperatures at one flat end (containing origin) and at the curved surface are prescribed. The initial temperature of the cylinder is zero. Accordingly, the conditions may be stated as

$$v(r, 0, t) = 0, \quad v(a, z, t) = V(z, t), \quad v(r, z, 0) = 0, \qquad \ldots \quad (2)$$

where a is radius of the cylinder. The conductivity K_z is taken as $K_z = \lambda(1-z^2)$.

2. Solution of the Problem

To solve the partial differential eq. (1) we use the well known Laplace transform, defined as (Churchill 1954)

$$L\{v(r, z, t)\} = \bar{v}(r, z) = \int^{\infty} \exp.(-pt)\, v(r, z, t) \mathrm{d}t, \quad \mathrm{Re}(p) > 0 \qquad \ldots \quad (3)$$

with the inversion formula

$$v(r, t, z) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \exp(pt)\, \bar{v}(r, z) \mathrm{d}p, \qquad \ldots \quad (4)$$

where $0 < \sigma < 1$. Also we know

$$L\left\{\frac{\partial}{\partial t} v(r, z, t)\right\} = pL\{v(r, z, t)\} - v(r, z, 0). \qquad (5)$$

Hence with the help of initial condition (2), we transform differential eq. (1) in the form

$$\frac{\partial^2 \bar{v}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{v}}{\partial r} + \frac{\lambda}{k}\frac{\partial}{\partial z}\left[(1-z^2)\frac{\partial \bar{v}}{\partial z}\right] \quad \frac{\rho c}{k} p\bar{v}, \qquad \ldots \quad (5)$$

with the conditions

$$L\{v(r, 0, t)\} = 0, \quad L\{v(a, z, t)\} = \bar{V}(z), \qquad \ldots \quad (7)$$

where $\bar{V}(z) = L\{V(z, t)\}$.

Now we make use of the Legendre transform, defined in eq. (7)

$$v_n(r) = \int_0^1 \bar{v}(r, z) P_{2n+1}(z) \mathrm{d}z, \qquad \ldots \quad (8)$$

and having the inversion formula

$$\bar{v}(r, z) = \sum_{n=0}^{\infty} (4n+3) v_n(r) P_{2n+1}(z). \qquad \ldots \quad (9)$$

We know that the Legendre polynomial $P_{2n+1}(z)$ is solution of the differential equation

$$\frac{d}{dz}\left[(1-z^2)\frac{dP_{2n+1}}{dz}\right]+(2n+1)(2n+2)P_{2n+1}(z) = 0. \tag{10}$$

Hence using eqs. (8) and (9), we get eq. (6) in the form

$$\frac{\partial^2 v_n}{\partial r^2}+\frac{1}{r}\frac{\partial v_n}{\partial r}-\xi_n^2 v_n = 0, \tag{11}$$

where

$$\xi_n^2 = \lambda/k(2n+1)(2n+2)+p/\beta, \quad \beta = k/\rho c.$$

Eq. (11) is well known Bessel's equation and its solution is given as

$$v_n = AI_0(\xi_n r)+BK_0(\xi_n r) \qquad \dots (12)$$

But as $r \to 0$, $K_0(\xi_n r) \to \infty$. Therefore,

$B = 0$ and from eq. (7) we get

$$A = V_n/I_0(\xi_n a),$$

where

$$P_{2n+1}(z)\overline{V}(z)\mathrm{d}z. \qquad \dots (13)$$

Substituting the value of A in eq. (12), using eqs. (9) and (4) we get

$$v(r,z,t) = \sum_{n=0}^{\infty}(4n+3)\mathrm{P}_{2n+1}(z)\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} V_n\frac{I_0(\xi_n r)}{I_0(\xi_n a)}\exp(pt)dp. \tag{14}$$

3. Particular Case

Now as a particular case, take

$$V(z,t) = \mu z$$

so that from eqs. (7), (14) and the integrals (Rainville 1957)

$$\int_0^1 x^\gamma P_{2k+1}(x)\mathrm{d}x = \frac{1}{4k+3}, \quad \gamma = 2k+1 \tag{15}$$

$$= 0, \qquad \gamma \neq 2k+1,$$

we get

$$V_n = \mu/3, \qquad n = 0$$

$$= 0, \qquad n = 1, 2, \dots.$$

Hence from eq. (14) we get

$$v(r,z,t) = \mu z\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{I_0(\xi_0 r)}{pI_0(\xi_0 a)}\exp(pt)\mathrm{d}p, \tag{16}$$

$$\xi_0 = [2+p/\beta]^{\frac{1}{2}} \text{ assuming } \lambda = k.$$

The zeros of $I_0(\xi_0 a)$ are

$$2+p/\beta = -\alpha_n^2, \qquad \ldots (17)$$

where α_n are roots of the equation

$$J_0(a\alpha) = 0. \qquad \ldots (18)$$

Hence evaluating the integral on the right hand side of eq. (16), with the help of residue method, we obtain

$$v(r,z,t) = \mu z \left[1 - \frac{2}{a} \sum_{n=1}^{\infty} \frac{\alpha_n \exp(-\beta(2+\alpha_n^2)t) J_0(r\alpha_n)}{(2+\alpha_n^2) J_1(a\alpha_n)}\right]. \qquad \ldots (19)$$

This gives temperature at anypoint of the cylinder at time t.

4. Numerical Results

This example is comparable to that of a thin hollow metallic cylindrical vessel filled with some insulating anisotropic material. As we know in most of the cases of insulating materials, such as wood, cork etc. $\beta = 0{\cdot}002$ (approx.) in F.P.S. system. In this example μ gives range and system of the temperature measurement. Hence, considering two cases $a = 0{\cdot}1$ cm and $a = 0{\cdot}2$ cm of the radius, we obtain following two forms of the solution

$$\begin{aligned} v(r,z,t)_{a=0\cdot 1} = \mu z[1 &- 1{\cdot}5941 \exp(-1{\cdot}1608t)\, J_0(24{\cdot}048r) + 1{\cdot}0640 \exp(-6{\cdot}0681t) \\ &\times J_0(55{\cdot}201r) - 0{\cdot}8559 \exp(-14{\cdot}9789t)\, J_0(86{\cdot}537r) + 0{\cdot}7374 \\ &\times \exp(-27{\cdot}8095t)\, J_0(117{\cdot}915r) - \ldots \end{aligned} \qquad \ldots (20)$$

$$\begin{aligned} v(r,z,t)_{a=0\cdot 2} = \mu z[1 &- 1{\cdot}5282 \exp(-0{\cdot}3028t)\, J_0(12{\cdot}024r) + 1{\cdot}0636 \exp(-1{\cdot}5264t) \\ &\times J_0(27{\cdot}600r) - 0{\cdot}8555 \exp(-3{\cdot}7468t)\, J_0(43{\cdot}268r) + 0{\cdot}7372 \\ &\times \exp(-6{\cdot}9542t)\, J_0(58{\cdot}957r) - 0{\cdot}6432 \exp(-11{\cdot}1492t) \\ &\times J_0(74{\cdot}654r) + 0{\cdot}5823 \exp(-16{\cdot}2302t) J_0(90{\cdot}355r) - \ldots . \end{aligned} \qquad \ldots (21)$$

The numerical values of v', where $v' = v(r,z,t)/\mu z$, for different r and t are given in table 1 and table 2.

Table 1

t \ r	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
1	0.4953	0.5047	0.5257	0.5596	0.6054	0.6609	0.7214	0.7723	0.8626	0.9326	0.9987
2	0.8682	0.8701	0.8757	0.8847	0.8968	0.9115	0.9274	0.9460	0.9643	0.9825	0.9997
3	0.9509	0.9516	0.9537	0.9570	0.9616	0.9670	0.9730	0.9799	0.9867	0.9935	0.9969
4	0.9844	0.9847	0.9853	0.9864	0.9878	0.9896	0.9914	0.9936	0.9958	0.9989	1.0000
5	0.9952	0.9952	0.9954	0.9958	0.9962	0.9968	0.9973	0.9980	0.9987	0.9994	1.0000
6	0.9994	0.9994	0.9994	0.9995	0.9996	0.9997	0.9997	0.9998	0.9998	0.9999	1.0000

Table 2

t \ r	0.00	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20
1	0.0289	0.0308	0.0443	0.0673	0.1087	0.1759	0.2725	0.4202	0.5963	0.7956	1.0000
2	0.1829	0.1940	0.2188	0.2641	0.3283	0.4107	0.5057	0.6248	0.7480	0.8753	1.0000
3	0.3694	0.3777	0.4027	0.4436	0.4991	0.5676	0.6430	0.7329	0.8226	0.9128	1.0000
4	0.5270	0.5336	0.5533	0.5853	0.6283	0.6806	0.7375	0.8052	0.8705	0.9365	1.0000
5	0.6479	0.6529	0.6679	0.6921	0.7245	0.7637	0.8061	0.8558	0.9047	0.9533	1.0000
6	0.7418	0.7454	0.7564	0.7743	0.7979	0.8267	0.8578	0.8942	0.9301	0.9657	1.0000
7	0.7929	0.7957	0.8045	0.8187	0.8378	0.8609	0.8859	0.9151	0.9439	0.9725	1.0000

To discuss the variation of $v(r, z, t)$ with respect to r, we consider centigrade system for temperature measurement and we take $\mu = 110°C$. Figure 1 and figure 2 give the variation of v $(= v(r, z, t)/z)$ for different values of time while the figure 3 and figure 4 show variation of $v(r, z, 1)$ for different values of z.

5. Conclusion

The variation of temperature is shown inside the cylinder for two values of radius i.e., $r = 0{\cdot}1$ and $r = 0{\cdot}2$. Figures 1 and 2 give the variations in any circular cross section of the cylinder for different values of time. It is clear from figure 1 that as the time increases the temperature rapidly acquires a constant value. Hence when the radius is small the radial flow becomes negligible after a short interval of time.

While in figure 2 this process is slow. So as we go on increasing the radius of the cylinder, the radial flow will become more and more significant even for pretty long durations of time.

Figures 3 and 4 exhibit temperature variation after a unit interval of time with respect to radius, at some specific circular cross sections. Apart from supporting the above facts these figures show that for smaller radius the variation of temperature from the-axis to the rim is not much, near the base, but it increases as we move towards the top of the cylinder. While in the second case the variation is fast even at the planes nearer to the base and it further increases with the position of the cross sectional planes.

The results obtained here are comparable with those established by experimental data and the nature of thermal conductivity which we have discussed here, closely resembles with many substances of practical use (Tye 1969).

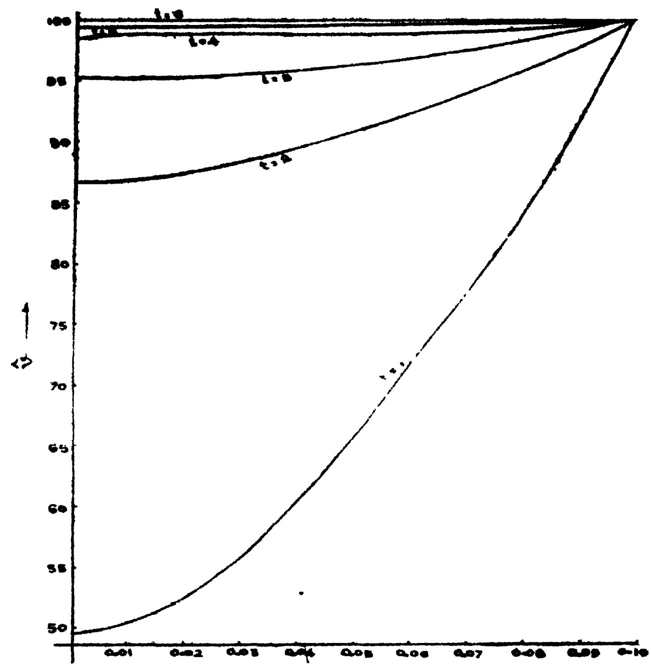

Figure 1. Graph between v and r ($a = 0{\cdot}1$, $\mu = 100$).

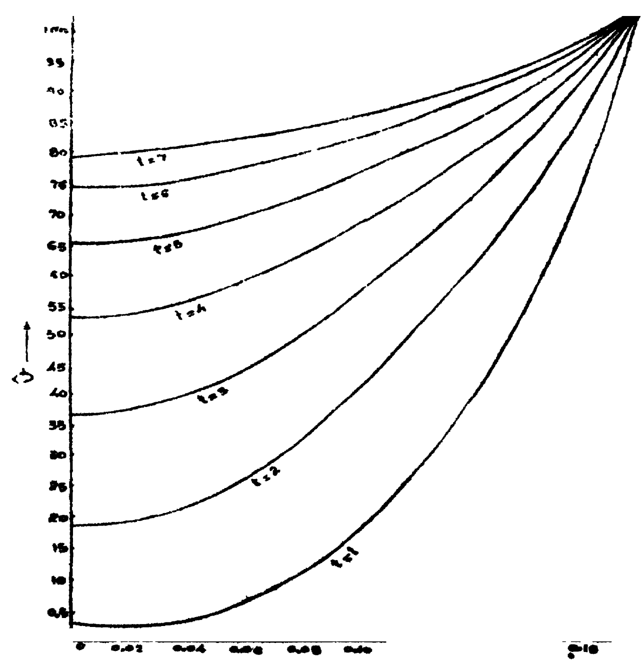

Figure 2. Graph between v and r ($a = 0{\cdot}2$, $\mu = 100$).

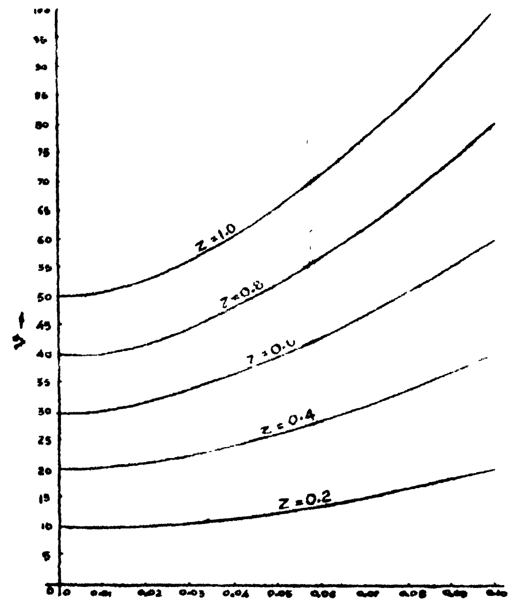

Figure 3. Graph between v and r ($a = 0{\cdot}1$, $\mu = 100$, $t = 1$)

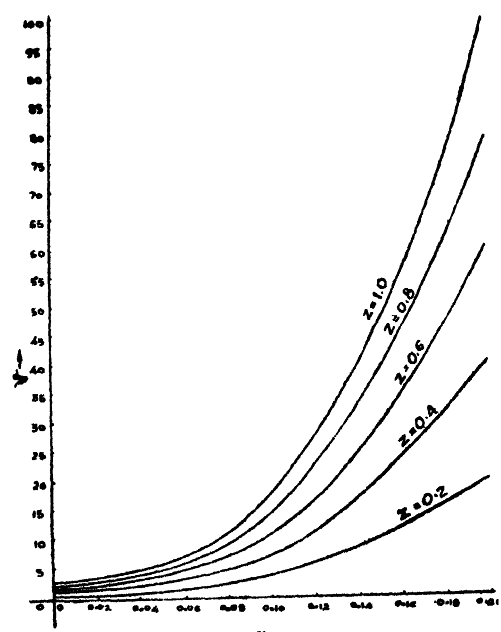

Figure 4. Graph between v and r ($a = 0{\cdot}2$, $\mu = 100$, $t = 1$).

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